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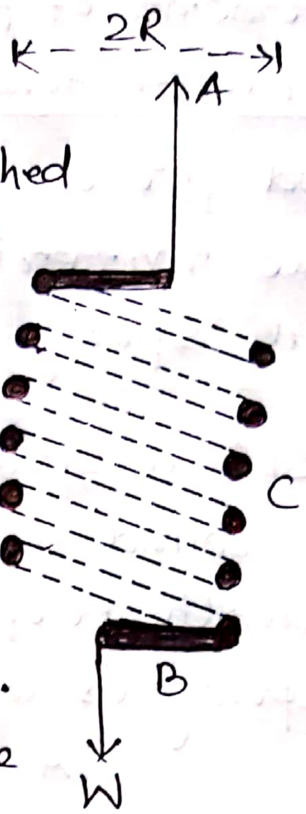
Method for determining the modulus of rigidity of the material in the form of a flat spiral spring.

Flat spiral Spring:- When the plane of wire wound round a circular cylinder is everewhere practically perpendicular to the axis of cylinder the spring is called the flat spiral spring.

A common spiral spring consists of a uniform wire of radius r wound round a circular cylinder of Radius R (R>>r) in such a way that the axis of the wire makes constant axis of the wire makes constant angle with the generating lines of the cylinder. When the spring is unstrained ~~is~~ the spring takes the form of a regular helix.

Experiment:- The top end is fixed at a point in line with the axis of the cylinder and a load is attached at the lower end which is also is a line with the axis of the cylinder A force and couple are assumed to act along the axis of the spring and in a plane perpendicular to it, even in the altered condition of the spring.

Theory! — Let the top end of the spring is fixed at a point A and a load W is attached to its free end B. Let us Consider the equilibrium of a portion CB. There must be a shearing force W acting vertically upward over this portion, and Couple of moment WR.



The moment of the Couple produces a uniform twist ϕ per unit length of the wire. The torsional rigidity for the unit length of the wire is

$$C = \frac{\pi \eta r^4}{2 \times 1} \cdot \phi$$

$$C = \frac{\pi \eta r^4 \phi}{2}$$

for equilibrium

$$WR = \frac{\pi \eta r^4 \phi}{2} \quad \text{--- (1)}$$

Let l be the total length of the wire

$\therefore$ Twist at the free end

$$B = l\phi$$

$\therefore$ work done in twisting the wire through an angle

$$\phi = dW = \frac{\pi \eta r^4 \phi . d\phi}{2l}$$

$\therefore$ Total work done in twisting the wire through $l\phi$ is

$$W' = \int_0^{l\phi} \frac{\pi \eta r^4 \phi \, d\phi}{2l}$$

$$= \frac{\pi \eta r^4}{2l} \left[\frac{\phi^2}{2}\right]_0^{l\phi}$$

$$= \frac{\pi \eta r^4}{2l} \cdot \left[\frac{l^2\phi^2}{2}\right]$$

$$\therefore W' = \frac{\pi \eta r^4 l \phi^2}{4} \quad \text{——— (2)}$$

This work is stored up as potential energy in the wire. Let the weight W is depressed by a distance x and then realeased. In this case the twist in the wire is increased by an amount ϕ_1 per unit length.

$$x = lR\phi_1$$

$\therefore$ Additional potential energy

$$W'' = \frac{\pi \eta r^4 l \phi_1^2}{4}$$

$$= \frac{\pi \eta r^4 l}{4} \left(\frac{x}{lR}\right)^2$$

$$W'' = \frac{\pi \eta r^4 x^2}{4lR^2} \quad \text{——— 3}$$

When W is ~~load~~ lowered by x, the centre of gravity of the Spring is lowered by $\frac{x}{2}$. Thus the chang in PE from these effects is

$$P.E = \frac{\pi\eta r^4 x^2}{4l R^2} - Wx - \frac{w_1 x}{2} \quad \text{--- (4)}$$

Where w_1 is the weight of the spring. The mass of the weight is $m = \frac{w}{g}$. on releasing w, its up and down.

$$\text{velocity} = \left(\frac{dx}{dt}\right)$$

$$\therefore \text{Kinetic energy of } w = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$$

$$K.E \text{ of } W = \frac{1}{2} \frac{W}{g} \left(\frac{dx}{dt}\right)^2 \quad \text{--- (5)}$$

The spring also possesses K.E. The vertical ~~end, measured along~~ ~~do des~~ depression at any ~~the~~ point in the spring is proportional to its distance and from the fixed end, measured along the length of the wire.

$$\text{The velocity at a distance } l = \left(\frac{dx}{dt}\right)$$

$$\text{velocity at a distance } s = \frac{s}{l} \left(\frac{dx}{dt}\right)$$

Let us consider an element of length ds at s. If μ be the mass per unit length, then the mass of the element $ds = \mu ds$

$$\therefore K.E. \text{ of the element is } = \frac{1}{2} \mu ds \left(\frac{s}{l}\frac{dx}{dt}\right)^2$$

$$\therefore K.E. \text{ of the whole spring is } = \int_0^l \frac{1}{2} \mu \left(\frac{s}{l}\frac{dx}{dt}\right)^2 ds$$

$$= \frac{\mu}{2l^2} \left(\frac{dx}{dt}\right)^2 \int_0^l s^2 ds$$

$$= \frac{\mu}{2l^2} \left(\frac{dx}{dt}\right)^2 \frac{l^3}{3}$$

$$= \frac{\mu l}{2 \times 3} \left(\frac{dx}{dt}\right)^2$$

$\therefore$ K.E of the whole spring $= \frac{W_1}{g \times 2 \times 3}\left(\frac{dx}{dt}\right)^2$ —— (6)

where $\frac{W_1}{g}$ is the mass of the spring

$\therefore$ Total K.E = K.E of W + KE of W_1

$$= \frac{1}{2}\frac{W}{g}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\frac{W_1}{3g}\left(\frac{dx}{dt}\right)^2$$

$$= \frac{1}{2g}\left(W + \frac{W_1}{3}\right)\left(\frac{dx}{dt}\right)^2 \quad \text{—— (7)}$$

From the principal of conservation of energy

$$KE + PE = \text{Constant}$$

$$\therefore \frac{1}{2g}\left(W + \frac{W_1}{3}\right)\left(\frac{dx}{dt}\right)^2 + \frac{\pi\eta r^4 x^2}{4l R^2} - Wx - \frac{W_1 x}{2} = \text{Constant}$$

Differentiating it with respect to t, we have

$$\frac{1}{2g}\left(W + \frac{W_1}{3}\right)\left(2\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + \frac{\pi\eta r^4 \times 2x}{4l R^2}\left(\frac{dx}{dt}\right) - \left(W + \frac{W_1}{2}\right)\frac{dx}{dt} = 0$$

$$\text{or, } \frac{1}{g}\left(W + \frac{W_1}{3}\right)\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + \frac{\pi\eta r^4 x}{2l R^2}\left(\frac{dx}{dt}\right) - \left(W + \frac{W_1}{2}\right)\left(\frac{dx}{dt}\right) = 0$$

Dividing by $\left(\frac{dx}{dt}\right)$, we have

$$\frac{d^2x}{dt^2} + \frac{\pi\eta r^4 x g}{2l R^2\left(W + \frac{W_1}{3}\right)} - g\left[\frac{W + \frac{W_1}{2}}{W + \frac{W_1}{3}}\right] = 0$$

$$\therefore \frac{d^2x}{dt^2} + \omega^2 x = \text{Constant} \quad \text{—— (8)}$$

where

$$\omega^2 = \frac{\pi\eta r^4 g}{2l R^2\left(W + \frac{W_1}{3}\right)}$$

This equation represents a simple harmonic motion. If T be the time period of oscillations of the spring, then

$$T = \frac{2\pi}{\omega}$$

$$\therefore T = 2\pi\sqrt{\frac{2lR^2\left(W + \frac{W_1}{3}\right)}{\pi\eta r^4 g}} \qquad (9)$$

Thus measuring the time period T, the modulus of rigidity η can be calculated from ~~eqn~~ equation (9).